Higher Order Mode Cutoff in Polygonal Transmission Lines

H. E. GREEN, SENIOR MEMBER, IEEE, AND J. D. CASHMAN, MEMBER, IEEE

Abstract —**This paper provides an analytical solution to the problem of determining the cutoff wavelength of the first higher order mode in any transmission line having as its conductors a pair of coaxial, similar, similarly oriented regular polygons. The method is based on the cross-sectional resonance technique. It is shown that, under certain circumstances, the requirement that the polygons be coaxial can be relaxed and a solution still obtained in this way.**

I. INTRODUCTION

In a recent paper, Gruner [1] has given the results of a numerical solution to the problem of determining cutoff wavelengths of the higher order modes in a square coaxial line. Gruner gives data for the first several higher order modes in graphical form.

From an engineering point of view, interest lies almost solely in ensuring that operation of the line is monomodal, i.e., only the dominant TEM mode propagates. Clearly then, most concern lies with determining the cutoff wavelength of the first higher order mode. Modes higher than this will generally be of little more than academic interest.

This note shows that cutoff of this mode can be determined by the cross-sectional resonance technique [2, p. 227]. The solution is analytic, except that the transcendental equation which finally determines the cutoff wavelength is best solved numerically. In the latter stages of deriving this equation, some approximation is involved but this is scarcely any disadvantage as extremely precise knowledge of the cutoff wavelength is not necessary if the aim is really with an adequate margin of safety to avoid the onset of higher order modes.

It will emerge that this technique is quite general. It is applicable to any transmission line made up of a pair of conductors which are coaxial, similar, similarly oriented regular polygons, of which a square coaxial line is but a special case. How the method might be used in yet more general cases will also be indicated.

II. THEORETICAL DEVELOPMENT

Consider the transmission line shown in cross section in Fig. 1. Although the figure shows a square coaxial line, the method applies to any line consisting of a pair of coaxial, similar, similarly oriented regular polygons.

To determine the cutoff wavelength of the first higher order mode, we need to know the conditions under which the cross section will go into resonance. We can regard the structure as a cascade of parallel-plate transmission lines joined by mitred bends and bent around to close on itself. Resonance can be determined by choosing a reference plane anywhere in one of the parallel-plate lines (or along the bisector of a bend), but the procedure is simplest if we choose the reference plane in one of the lines at its intersection with one of the principal axes of cross-sectional symmetry, such as AA in Fig. 1.

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The authors are with the Department of Electrical and Electronic Engineering, University of New South Wales, Faculty of Military Studies, Royal Military College, Duntroon, ACT 2600, Australia.

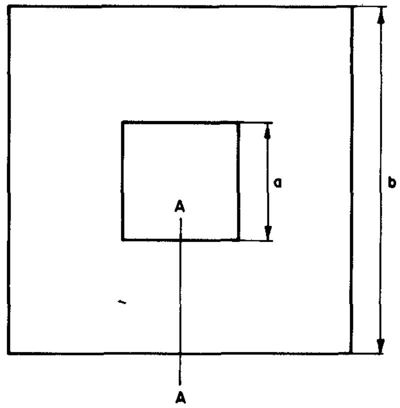

Fig. 1. Coaxial transmission line of a square cross section. At resonance, admittance is zero at reference plane AA.

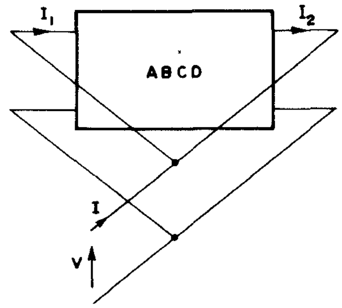

Fig. 2. Equivalent circuit for transmission around the structure of Fig. 1.

Consider unit axial length of the line. The condition for resonance is that the admittance presented at AA should be zero. Fig. 2 shows an equivalent circuit representation for the system, the "black box" representing the entire cascade of lines and bends. Using a transmission (ABCD) matrix representation [2, p. 85] of the cascade, it is easy to show that resonance corresponds to

$$A + D = 2. \tag{1}$$

Since AA is chosen to be on one of the axes of symmetry, $A = D$ and (1) simplifies to

$$A = 1. \tag{2}$$

For n-sided polygons, the black box consists of a cascade of n identical subnetworks, each containing one corner. Fig. 3(a) shows one of these subnetworks for the case of a square coaxial line. It consists of two parallel-plate lines of length $a/2$ joined by a corner region. In any other case, the network will be similar except that the angle of the mitre will no longer be $\pi/2$ but $2\pi/n$. The transmission matrix $[T]$ of the cascade will be

$$[T] = [t]^n \tag{3}$$

where

$$[t] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4}$$

is the transmission matrix of one of the subnetworks. It is easily shown, e.g., by the Cayley–Hamilton theorem [3, p. 1097], that

$$A = \cos\{ n(\cos^{-1} a)\} \tag{5}$$

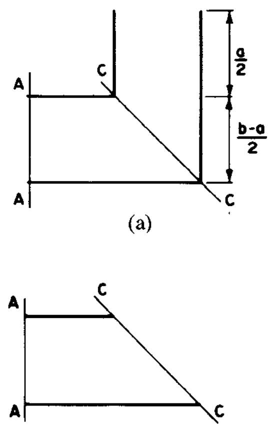

Fig. 3. (a) One of the n bends which constitute the structure of Fig. 1. (b) One of the two halves of the bend.

and so the condition for cross-sectional resonance is that

$$a = \cos\frac{2\pi}{n}. \tag{6}$$

Moreover, as Fig. 3 shows, each of these subnetworks is symmetrical about the bisector of the mitre. It is a cascade of two identical, asymmetric networks connected back-to-back and it is simple to show that

$$a = 2\alpha\beta - 1 \tag{7}$$

where $[\tau]=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is the transmission matrix of the network shown in Fig. 3(b), a section of transmission line followed by a half corner. Combining (6) and (7) then allows us to deduce that the condition for resonance must be

$$\alpha\beta = \cos^2\frac{\pi}{n}. \tag{8}$$

Equation (8) is accurate, but to proceed further requires approximation. We will assume that if the bend region is sufficiently small—exactly what this means is a question that can be deferred temporarily—it can be replaced by a Π network of series inductance and shunt capacity, i.e., we are assuming that Fig. 3(b) can be represented circuitally by the equivalent network shown in Fig. 4. L is the total inductance of the complete corner region and C is the capacitance.

The transmission matrix $[\tau]$ for this cascade is easily found by multiplication of the transmission matrices of each subnetwork in the cascade. When the result is inserted into (8), the condition for cross-sectional resonance becomes

$$\cos^2\frac{\phi}{2}\left(1-\frac{1}{2}Z_0C\omega\tan\frac{\phi}{2}\right)\cdot\left\{\left(1-\frac{1}{4}LC\omega^2\right)-\frac{1}{2}\frac{L\omega}{Z_0}\tan\frac{\phi}{2}\right\}-\cos^2\frac{\pi}{n}=0 \tag{9}$$

where $C(L)$ is the total capacitance (inductance) attributable to the mitre region,

$\phi \quad = \dfrac{2\pi a}{\lambda}$,
$\omega \quad = 2\pi f$,
f is the frequency, and
λ is the wavelength corresponding to f.

The solution of (9) gives the condition for the first higher order mode cutoff. All that is required to solve this equation are values for the parameters Z_0, L, and C.

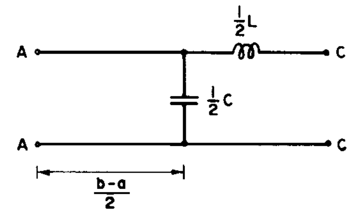

Fig. 4. Equivalent circuit for Fig. 3(b).

Analytic expressions from which they can be obtained are given in the *Waveguide Handbook* [4]. The normalized susceptances of the series and shunt elements of the Π equivalent circuit of an E-plane bend in a parallel-plate line will be the same as those for an E-plane bend in a rectangular waveguide [2, p. 347–348], i.e. results for $Z_0C\omega$ and $Z_0/L\omega$ are directly available.

To apply this data to this problem, it is useful to rework it in terms of the mean line circumference

$$C_m = \frac{n(a+b)}{2} \tag{10}$$

and the flat width ratio

$$s = \frac{b}{a}. \tag{11}$$

After some amount of not very interesting geometry, we obtain

$$Z_0C\omega = \frac{4(s-1)\Sigma_n}{n^2\Lambda(s+1)\tan\frac{\pi}{n}} \tag{12}$$

$$\frac{L\omega}{Z_0} = \frac{2\pi(s-1)}{n\Lambda(s+1)} \tag{13}$$

where $\Lambda = \lambda/C_m$ is the normalized wavelength, and

$$\Sigma_n = \sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{2}\right)\left(k+\frac{1}{2}+\frac{1}{n}\right)}.$$

When these results are substituted into (9), we obtain an equation for the normalized cutoff wavelength as a function of s with n as a parameter as follows:

$$\cos^2\frac{2\pi}{n(s+1)\Lambda_c}\left(1-\frac{2(s-1)\Sigma_n}{n^2\Lambda_c(s+1)\tan\frac{\pi}{n}}\tan\frac{2\pi}{n(s+1)\Lambda_c}\right)\cdot\left(1-\frac{2\pi(s-1)^2\Sigma_n}{n^3\Lambda_c^2(s+1)^2\tan\frac{\pi}{n}}-\frac{\pi(s-1)}{n\Lambda_c(s+1)}\tan\frac{2\pi}{n(s+1)\Lambda_c}\right)-\cos^2\frac{\pi}{n}=0. \tag{14}$$

Obviously, this is best solved numerically. Examination of this equation shows that $2/(s+1)<\Lambda_c<1$, i.e., the cutoff wavelength is always less than the mean circumference, except for the trivial case $s=1$, where they are equal.

One is entitled to ask, of course, the conditions under which the approximations involved in deriving (14) will be fulfilled. There are two conditions.

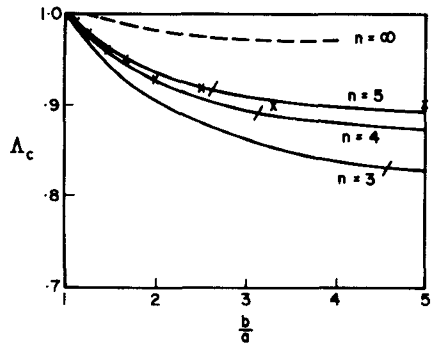

Fig. 5. Normalized cutoff wavelength as function of flat-width ratio. Slashes indicate maximum values of $\frac{b}{a}$ for which theory is expected to be valid. Curve $n = \infty$ is circular case. Crosses are Gruner's results [1] for $n = 4$.

The first requires that the corner regions must be small enough compared with the wavelength that the quasistatic assumptions used to derive the equivalent network constants L and C are justified. Given that we observe that $\Lambda_c \approx 1$, this must always be so.

The other is that the corners are far enough apart so that each can be considered isolated from the others. Essentially this means that a, the length of each of the parallel-plate line segments, should not become much less than comparable with the separation of the plates [5]. The greatest value of s for which the solution is likely to be accurate is $s \approx 1 + 2\tan \pi/n$. For a square coaxial line ($n = 4$), this would give $s \approx 3$ and for an air-filled square coaxial line where $s = 3.5$, $Z_0 \cong 70\Omega$ [6]. It is reasonable to conclude that, for this case, the method is valid for almost all lines likely to be of any practical significance.

In Fig. 5 are shown results computed by this method for $n = 3$, 4, and 5. The slashes indicate the maximum values of s for which the theory would ordinarily be believed to be good. The crosses are sample values from Gruner's numerical solution for the square coaxial line, and are to be compared with our curve for $n = 4$. Agreement is seen to be very good well beyond the range in which the theory is expected to hold. The curve labelled $n = \infty$, the circular coaxial line, is included for comparison, although the present theory is not applicable to it.

III. Generalization of the Method

This technique can be applied to more general cases, such as a line consisting of a rectangle within a rectangle. Even concentricity is not required; all that is needed is that the cross section be made up of sections of parallel-plate line joined by mitred elbows. In this more general case, simplications which result from symmetry are, of course, no longer available. Equation (1) needs to be used to determine resonance and a large number of different matrices will have to be multiplied to determine A, D.

Determination of L and C for each corner could still be undertaken quasistatically on the assumption of isolated corners. In the more general case for L this is easy [7], but for C resorting to some numerical technique such as finite differences would be needed [5]. Valid application of the method continues to rest on having a cross section with small, well-isolated corners.

It may be true though that—unless one enjoys advantages such as the ready availability of a software package for handling finite-difference solutions of Laplace's equation—for these more general cases, if a precise answer is required the cross-sectional resonance technique begins to lose its advantage over a purely numerical solution. On the other hand, if a bound on the answer is all that is required, this method would indicate that a good opening approximation is simply to assume that the cutoff wavelength equates to the mean line circumference.

IV. Conclusions

A theoretical development has been given which allows approximate determination of the cutoff wavelength of the first higher order mode in any transmission line consisting of a pair of coaxial, similar, similarly oriented regular polygons. Comparison for the case of a square coaxial line with results obtained by a purely numerical method indicates that agreement within a few percent is to be expected for all lines having characteristic impedances likely to be of practical interest. Moreover, even without solving the transcendental equation which this approach produces, it is possible to put bounds on the normalized cutoff wavelength of the first higher order mode. If the problem is simply to avoid exciting it, this alone may be enough. It has also been shown that this method is capable of handling more general problems that do not exhibit a high degree of symmetry.

Acknowledgment

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An Explicit Six-Port Calibration Method using Five Standards

J. D. HUNTER, SENIOR MEMBER, IEEE, AND
P. I. SOMLO, SENIOR MEMBER, IEEE

***Abstract*—A six-port reflectometer calibration method using five standards is developed, and gives explicit unambiguous expressions for the calibration constants. The standards are restricted only in that their impedances may neither all have the same magnitude nor all have the same**

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The authors are with the CSIRO Division of Applied Physics, P.O. Box 218, Lindfield, N.S.W. 2070, Australia.

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